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Exact interface model for wetting in the planar Ising model

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At the wetting transition in the two-dimensional Ising model the long contour (interface) gets depinned from the substrate. It is found that on sufficiently *large* length scales the statistics of the long contour are described by a *unique* probability measure corresponding to a continuous ''interface model'' with an interface binding ''potential'' given by a Dirac δ function supported on the substrate. A lattice solid-on-solid model is shown to give similar results. [S1063-651X(99)51010-6]

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In this Rapid Communication we address the theory of wetting in systems with short-ranged forces as exemplified by the Ising model. A mean-field description of wetting was first obtained from Landau theory by Cahn [1] and developed further by Nakanishi and Fisher [2]. Abraham [3] studied critical wetting in a two-dimensional Ising model using exact methods and found behavior at temperatures close to the wetting temperature, T_w, different from that predicted by mean field theory. The situation in three dimensions is far less clear since fluctuation effects are still believed to be important. Progress was thought possible by coarse-graining to sufficiently large length scales, such as the *bulk* correlation length, $\xi_{\rm b}$, so that bulklike fluctuations can be ignored and one need consider only the interfacial degrees of freedom of the wetting layer. Thus, for a d-dimensional system, the resulting *interface model* is then expressed in terms of the substrate-interface separation, $y(\mathbf{x})\geqslant 0$, above a point $\mathbf{x}$ in the substrate $S\subset\mathbb{R}^{d-1}$ for which one assigns the following effective Hamiltonian, $\mathcal{H}_{\rm eff}[y]$ [4],

$$\mathcal{H}_{\rm eff}[y]=\int_S d\mathbf{x}\left[\frac{1}{2}\tilde{\tau}|\boldsymbol{\nabla} y|^2+V(y)\right], \tag{1}$$

where $\tilde{\tau}$ is the interfacial stiffness. The interfacial potential, $V(y)$, was originally given the form [4,5]

$$V(y)=v_1(T)e^{-y/\xi_{\rm b}}+v_2e^{-2y/\xi_{\rm b}}, \tag{2}$$

where v_2 is taken to be positive and independent of temperature T and $v_1(T)\propto T-T_{\rm w}^{\text{mean-field}}$. A more recent and careful derivation of $V(y)$, starting from a bulk Landau-Ginzburg-Wilson Hamiltonian, has been carried out [6] which leads to Eq. (2) together with corrections (see also [7]). The partition function, Z_S, is then written in terms of the functional integral

$$Z_S=\prod_{\mathbf{x}\in S}\int_0^\infty dy(\mathbf{x})e^{-\mathcal{H}_{\rm eff}[y]}, \tag{3}$$

although this is only a formal expression whose precise mathematical meaning is unclear at this stage and implicit within it is some lower-length cutoff.

A more rigorous approach to such interface models is still lacking so it seems timely to clarify the situation as it applies to the $d=2$ Ising model. In this case Eq. (3), with Eq. (1), is analogous to the evolution kernel for Brownian motion (with diffusion constant $1/2\tilde{\tau}$) on the half line $y\geqslant 0$ subject to a potential $V(y)$ with x being timelike. The main result reported here is that *exact* analysis of the planar Ising model *and* a lattice solid-on-solid (SOS) model implies that $V(y)$ for these models is given by $V(y)=c\,\delta_0(y)$ where $\delta_0(\cdot)$ is the Dirac δ distribution supported on $\{0\}$ and c can be expressed in terms of the temperature and various microscopic parameters. This result applies only to length scales sufficiently larger than the bulk correlation length, $\xi_{\rm b}$, but on these scales Eq. (2) converges in some sense to a δ function [8]. Otherwise, Eqs. (1) and (3) still *formally* hold and for the Ising model, $\tilde{\tau}$ is the surface stiffness [9] which takes into account lattice anisotropy. At the critical wetting transition $c=0$ with $c>0$ (respectively, $c<0$) for $T>T_w$ (respectively, $T<T_w$). Before presenting the main result in a more precise way, we first describe the two microscopic lattice models considered.

Planar Ising Model. Ising spins, $\sigma_{m,n}=\pm 1$ ($1\leqslant m\leqslant M$, $1\leqslant n\leqslant N$), are placed on the sites of a two-dimensional square lattice wrapped onto a cylinder of height N and circumference M. The Ising spins interact across nearest-neighbor sites with couplings K_1 (respectively, K_2) in the (0,1) [respectively (1,0)] direction ($K_j=J_j/k_{\rm B}T$). Following [3], two types of boundary conditions are imposed along the bottom edge of cylinder by adding an extra row of spins along $\{(m,0)\}_{m=1}^M$ coupled to the row $\{(m,1)\}_{m=1}^M$ by vertical bonds of strength h_1. In case $\mathcal{A}$ one fixes $\sigma_{m,0}=+1$ for *all* $1\leqslant m\leqslant M$; in case $\mathcal{B}$, $\sigma_{m,0}=-1$ for $1\leqslant m\leqslant x$ and $\sigma_{m,0}=+1$ otherwise. The coupling h_1 acts as a surface field on the row $\{(m,1)\}_1^M$. The top edge of the cylinder is left free. In contrast to $\mathcal{A}$, boundary condition $\mathcal{B}$ induces a *long contour* (i.e., the interface) joining $(\frac{1}{2},\frac{1}{2})$ to $(x+\frac{1}{2},\frac{1}{2})$ on the dual lattice. In case $\mathcal{B}$, Abraham [3] showed that after taking the limits $M\to\infty$, $N\to\infty$ followed by $x\to\infty$ a wetting transition occurs. If w is defined by

$$w:=e^{2K_2}(\cosh 2K_1-\cosh 2h_1)/\sinh 2K_1, \tag{4}$$

then the interface stays pinned to the substrate (i.e., the bottom edge) when $w>1$ and depinned when $w<1$ with a critical wetting transition occurring at $w=1$. This shows up as a singularity in the *incremental free energy* for the interface defined as

$$\tau^{\times} := -\lim_{x\to\infty}\lim_{N\to\infty}\lim_{M\to\infty}\frac{1}{x}\ln[Z^{\mathcal{B}}/Z^{\mathcal{A}}], \tag{5}$$

where $Z^{\mathfrak{b}}$ is the canonical partition function for boundary condition $\mathfrak{b}=\mathcal{A},\mathcal{B}$.

SOS Model. We consider the model for an interface defined on a *lattice* as introduced by Abraham and Smith [10]. The interface is represented by the Markov random field $Y=(Y_j)_{j=0}^{x}$, where $Y_j\in[0,\infty)$ is the height of the interface above the lattice point j. The Gibbs measure, $\mathbf{Q}_x(\cdot)$, is then given by

$$\mathbf{Q}_x(Y\in dy)=\frac{1}{Z_x}\exp\left(-\kappa\sum_{j=1}^{x}|y_j-y_{j-1}|\right)\times\left[\prod_{j=1}^{x-1}(1+a\,\delta_0)(dy_j)\right]\times\delta_0(dy_0)\,\delta_0(dy_x), \tag{6}$$

where Z_x is the canonical partition function defined so that $\mathbf{Q}_x(\cdot)$ normalizes to 1. It was found that, in the limit $x\to\infty$, this model has a wetting transition at $a=1/\kappa$ with the interface being pinned (respectively, depinned) to the substrate for $a>1/\kappa$ (respectively, $a<1/\kappa$) [10].

In both cases, we seek the probability measure that describes the statistics of the interface on sufficiently *large* length scales. First, consider the tied-down Brownian motion [11] $(B_s)_{s\in[x_1,x_2]}$ on $\mathbb{R}$ with $B_{x_1}=y_1$, $B_{x_2}=y_2$ (which has diffusion constant $1/2\tilde{\tau}$). Then the tied-down *reflected* Brownian motion is formed by $(|B_s|)_{s\in[x_1,x_2]}$ for which we assign a conditional measure $\mu_{(x_1,y_1)}^{(x_2,y_2)}$ with the normalization $\int d\mu_{(x_1,y_1)}^{(x_2,y_2)}=g_-+g_+$ with $g_\pm=g(x_2-x_1;y_2\pm y_1)$ where $g(x;y)$ is the Gauss kernel $g(x;y)=(\tilde{\tau}/2\pi x)^{1/2}e^{-\tilde{\tau}y^2/2x}$. Now let $(Y_s\in[0,\infty))_{s\in[0,x]}$ represent the height of the interface above the substrate on a *sufficiently large* length scale. We show that for both the Ising and SOS case, its probability measure, $\mathbf{P}_x^c$, on the infinite-dimensional space $\Omega_x=[0,\infty)^{[0,x]}$, is given by

$$\mathbf{P}_x^c(\cdot)=\frac{1}{Z_x(c)}e^{-2cL_x}\mu_{(0,0)}^{(x,0)}(\cdot), \tag{7}$$

where the partition function, $Z_x(c)$, is

$$Z_x(c)=\int d\mu_{(0,0)}^{(x,0)}e^{-2cL_x} \tag{8}$$

and the random variable L_x is the Brownian "local time" [11] defined by

$$L_x:=\lim_{\epsilon\downarrow 0}\frac{1}{4\epsilon}\text{meas}\{0\leq s\leq x: Y_s\leq\epsilon\} \tag{9}$$

with meas$\{\cdot\}$ denoting the Lebesgue measure. Note that $\mathbf{P}_x^{c=0}$ is the probability measure for the tied-down reflected Brownian motion. The random variable L_x is a measure of the amount of substrate which stays close to the interface. Formally, it can be expressed as $2L_x=\int_0^x\delta_0(Y_s)ds$ in which case Eqs. (7) and (8) can be regarded as describing a Brownian particle (or Schrödinger particle of mass $\tilde{\tau}$) on the half line $y\geq 0$ subject to the "potential" $c\,\delta_0(y)$. For the Ising model $c=(1-w)/2\tilde{\tau}$, where the interfacial stiffness [9] is $\tilde{\tau}=\sinh 2K_1^*\sinh 2K_2\sinh\tau$ with τ being interfacial tension of a free interface given by $\tau=2(K_1-K_2^*)$ and $e^{-2K_j^*}=\tanh K_j$. For the SOS model $c=(1/\kappa)-a$ and $\tilde{\tau}=\kappa^2/2$. The incremental free energy (5) is now given by

$$\tau^{\times}(c)-\tau=-\lim_{x\to\infty}\frac{1}{x}\ln Z_x(c). \tag{10}$$

We stress that the process $(Y_s)_{s\in[0,x]}$ determines the properties of the Ising interface on length scales sufficiently larger than the bulk correlation length $\xi_b=1/2\tau$. For $T<T_w$, the wetting-layer thickness, ℓ, given by the expectation $\lim_{x\to\infty}\mathbf{E}Y_{x/2}=\ell$, should also be sufficiently larger than ξ_b. Therefore, for $T<T_w$, this description is strictly only appropriate for T sufficiently close to T_w (i.e., $w-1$ sufficiently small when positive and similarly for $a-1/\kappa$ in the SOS model). For $T>T_w$, the only restriction on T is that it be less than the bulk critical temperature. Thus, the process $(Y_s)_{s\in[0,x]}$, with measure $\mathbf{P}_x^c$, determines the *asymptotic* properties of wetting in the *scaling* regime.

We now outline how we arrived at these results. First, consider the family of finite-dimensional distributions on $[0,\infty)^n$ for all $n\geq 1$

$$\mathbf{P}_x^c(\Omega_x|Y_{x_1}\in A_1,\ldots,Y_{x_n}\in A_n)=\int_{A_1}dy_1\ldots\int_{A_n}dy_n\;p_{x,n}(x_1,y_1;\ldots;x_n,y_n), \tag{11}$$

where $0<x_1<\cdots<x_n<x$, $A_j\subset[0,\infty)$ for $j=1,\ldots,n$ and $p_{x,n}(\cdot)$ is the joint probability density function which, according to Eqs. (7) and (8), is given by

$$p_{x,n}(x_1,y_1;\ldots;x_n,y_n)=\frac{K(x_1;0,y_1)K(x-x_n;y_n,0)}{K(x;0,0)}\times\prod_{j=2}^{n}K(x_j-x_{j-1};y_{j-1},y_j), \tag{12}$$

with $K(\cdot)$ defined by

$$K(u;y_0,y):=\int d\mu_{(0,y_0)}^{(u,y)}e^{-2cL_u}. \tag{13}$$

It will prove useful to express $K(\cdot)$ more explicitly by use of a Feynman-Kac formula. This is done by applying Dirichlet-form techniques [12] from which it follows that $K(u;y_0,y)$ is the kernel of the evolution operator $e^{-u\hat{H}_c}$ on the half line with generator $\hat{H}_c=\hat{H}_0+c\,\delta_0$ where $\hat{H}_0$ is the Neumann-boundary-condition operator $\hat{H}_0\psi=-\psi''(y)/2\tilde{\tau}$ for $y>0$ with $\psi'(0)=0$. The term $c\,\delta_0$ in $\hat{H}_c$ can be treated as a rank-1 perturbation on $\hat{H}_0$ and, hence, it can be shown that $\hat{H}_c$ is equivalent to $-(2\tilde{\tau})^{-1}d^2/dy^2$ on $(0,\infty)$ with the

c-dependent boundary condition $\psi'(0)=2\tilde{\tau}c\psi(0)$ [13]. From this, $K(\cdot)$ can be expressed in spectral form [14]

$$K(u;y_0,y)=\Theta(-c)4\tilde{\tau}|c|e^{2\tilde{\tau}c^2u}e^{-2\tilde{\tau}|c|(y_0+y)}+\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-\omega^2u/2\tilde{\tau}}\left[e^{i\omega(y_0-y)}-\left(\frac{2\tilde{\tau}c+i\omega}{2\tilde{\tau}c-i\omega}\right)e^{i\omega(y_0+y)}\right], \tag{14}$$

where $\Theta(\cdot)$ is the Heaviside step function.

To evaluate $p_{x,n}(\cdot)$ for the planar Ising model we start by considering joint probabilities of lattice-contour events. Let $\mathcal{E}_{m,n}:=\sigma_{m,n}\sigma_{m+1,n}$ be the (horizontal) bond energy. Then $I_{m,n}:=(1-\mathcal{E}_{m,n})/2$ is the indicator for a Peierls contour vertically crossing the bond joining (m,n) to $(m+1,n)$. If j denotes the lattice site (x_j,y_j), $X:=\{1,\ldots,n\}$, $\mathcal{E}^X:=\prod_{j\in X}\mathcal{E}_j$ and $I^X:=\prod_{j\in X}I_j$, then the *joint* probability of Peierls contours vertically crossing bonds at $\{(x_j+\frac{1}{2},y_j)\}_{j\in X}$, with boundary condition $\mathfrak{b}$, is given by the canonical expectation $\langle I^X\rangle_{\mathfrak{b}}$ (where, throughout, the limits $N,M\to\infty$ have already been taken). In the presence of a long contour, the probability $\langle I^X\rangle_{\mathcal{B}}$ can be shown to be given by

$$\langle I^X\rangle_{\mathcal{B}}=\sum_{X'\subseteq X}(-1/2)^{|X'|}\langle\mathcal{E}^{X'}\rangle_{\mathcal{B}}^{\rm con}\langle I^{X\setminus X'}\rangle_{\mathcal{A}}, \tag{15}$$

where $|X'|$ is the cardinality of the set X' and the sum includes the empty set, $\emptyset$, with the convention $I^{\emptyset}=\mathcal{E}^{\emptyset}=1$. Also, $\langle\mathcal{E}^{X'}\rangle_{\mathcal{B}}^{\rm con}$ is the *connected* $|X'|$-point bond-energy correlation function truncated so that $\langle\mathcal{E}^{X'}\rangle_{\mathcal{B}}^{\rm con}\to 0$ whenever $\max\{|x_j|,|x_j-x|\}_{j\in X'}\to\infty$. Unlike $\langle I^X\rangle_{\mathcal{B}}$, the joint probability $\langle I^X\rangle_{\mathcal{A}}$ is translationally invariant in the x direction, i.e., invariant under $\{x_j\}_{\forall j\in X}\mapsto\{x_j+u\}_{\forall j\in X}$, and can be written

$$\langle I^X\rangle_{\mathcal{A}}=\sum_{\varpi\in\mathfrak{P}}\prod_{P\in\varpi}\langle I^P\rangle_{\mathcal{A}}^T, \tag{16}$$

where $\mathfrak{P}$ is the set of all *partitions* of X, $\varpi=\{P_1,\ldots,P_{|\varpi|}\}$ is an element of $\mathfrak{P}$, P is an element of ϖ of $\mathfrak{P}$, and $\langle I^P\rangle_{\mathcal{A}}^T$ denotes the *truncated* function which can be expressed in terms of the truncated $|P|$-point bond-energy correlation function as $\langle I^P\rangle_{\mathcal{A}}^T=(-1/2)^{|P|}\langle\mathcal{E}^P\rangle_{\mathcal{A}}^T$. The joint probability $\langle I^X\rangle_{\mathcal{B}}$ contains contributions coming from the *long contour* passing through *all*, *some* or *none* of the points in X with closed cycles, *disconnected* from the long contour, passing through the remaining points. If the points in X are sufficiently *well separated* then the terms in Eq. (15) can be understood as follows: $(-1/2)^{|X'|}\langle\mathcal{E}^{X'}\rangle_{\mathcal{B}}^{\rm con}$ is the probability (up to an unimportant prefactor) of the long contour passing through *all* the points in $X'\subseteq X$ whereas $\langle I^{X\setminus X'}\rangle_{\mathcal{A}}$ is the probability of contours *disconnected* from the long contour passing through the points in $X\setminus X'$. This identification is clear from the truncation properties of $\langle\mathcal{E}^{X'}\rangle_{\mathcal{B}}^{\rm con}$ and the translational invariance of $\langle I^{X\setminus X'}\rangle_{\mathcal{A}}$ (which is dominated by small bulklike bubbles passing through the points in $X\setminus X'$). Furthermore, $\langle I^P\rangle_{\mathcal{A}}^T$ in Eq. (16) is dominated by the probability of a *single* closed contour passing through *all* the points in P from which one can extract a large-deviations rate functional of Wulff type.

The truncated n-point bond-energy correlation functions, $\langle\mathcal{E}^X\rangle_{\mathcal{A}}^T$ and $\langle\mathcal{E}^X\rangle_{\mathcal{B}}^{\rm con}$, can be evaluated using transfer-matrix methods, results for which have already been presented in terms of Pfaffians of dimension $2n$ and $2n+2$ for cases $\mathcal{A}$ and $\mathcal{B}$, respectively [15]. Here we re-express these Pfaffians more transparently as follows. Let $\Gamma=\{(i_1,i_2),(i_2,i_3),\ldots,(i_{n-1},i_n)\}$ be the path defined in terms of the sequence of line elements where $\{i_1,\ldots,i_n\}$ is some *permutation* of X. A closed circuit, Γ_c, can be formed by $\Gamma_c=\{\Gamma,(i_n,i_1)\}$. For each line element (j,k) (i.e., pair of points in X) we assign a matrix

$$\mathbf{G}_{jk}=\begin{pmatrix}-G_{jk}^{-+} & G_{jk}^{--}\\ -G_{jk}^{++} & G_{jk}^{+-}\end{pmatrix} \tag{17}$$

if $y_k\geqslant y_j$ and $\mathbf{G}_{jk}$ is as in Eq. (17) for $y_k<y_j$ except that the superscripts $++$ and $--$ are interchanged for both the off-diagonal elements. The matrix elements are given by

$$G_{jk}^{pq}=\frac{q}{2\pi}\int_{-\pi}^{\pi}d\omega\; e^{i(x_k-x_j)\omega}e^{i(p-q)\delta^*(\omega)/2}\left[e^{-|y_k-y_j|\gamma(\omega)}-ip\frac{A^-(\omega)}{A^+(\omega)}e^{-(y_j+y_k)\gamma(\omega)}\right] \tag{18}$$

(for $p,q\in\{+,-\}$), where

$$A^{\pm}(\omega)=\frac{e^{-K_2}\sinh 2K_1}{\sinh 2h_1}[e^{\pm\gamma(\omega)}-w]\begin{cases}\cos[\delta^*(\omega)/2]\\ \sin[\delta^*(\omega)/2]\end{cases} \tag{19}$$

and $\gamma(\omega)$ and $\delta^*(\omega)$ are the Onsager functions defined in terms of a length and angle, respectively, of a hyperbolic triangle and can be found in, e.g., Ref. [15]. Thus, if $\mathbf{G}^{\Gamma}:=\mathbf{G}_{i_1i_2}\mathbf{G}_{i_2i_3}\cdots\mathbf{G}_{i_{n-1}i_n}$ and $\mathbf{G}^{\Gamma_c}:=\mathbf{G}^{\Gamma}\mathbf{G}_{i_ni_1}$, then the required bond-energy correlation functions can be expressed as

$$\langle\mathcal{E}^X\rangle_{\mathcal{A}}^T=\frac{-1}{2}\sum_{\Gamma_c}\mathrm{Tr}\,\mathbf{G}^{\Gamma_c}, \tag{20}$$

$$\langle\mathcal{E}^X\rangle_{\mathcal{B}}^{\rm con}=\sum_{\Gamma}U_{i_1}^t\mathbf{G}^{\Gamma}V_{i_n}/G(x), \tag{21}$$

where the sums are over all distinct circuits and paths, $G(x)$ is given by

$$G(x)=\frac{1}{2\pi i}\int_{-\pi}^{\pi}d\omega\frac{B(\omega)e^{ix\omega}}{A^+(\omega)}, \tag{22}$$

where

$$B(\omega)=\frac{e^{K_2}\sinh 2K_1}{\sinh 2h_1}[e^{\gamma(\omega)}-e^{-4K_2}w]\sin[\delta^*(\omega)/2] \tag{23}$$

and the initial and final vectors in Eq. (21) are given by $U_{i_1}^t=[G_{i_1}^+(0),-G_{i_1}^-(0)]$, $V_{i_n}^t=[G_{i_n}^-(x),G_{i_n}^+(x)]$, with

$$G_j^{\pm}(u)=\frac{\pm 1}{2\pi}\int_{-\pi}^{\pi}d\omega\,\frac{e^{\pm i\delta^*(\omega)/2}}{A^+(\omega)}e^{i(u-x_j)\omega}e^{-y_j\gamma(\omega)}. \tag{24}$$

We seek the form of $\langle\mathcal{E}^X\rangle_{\mathcal{B}}^{\mathrm{con}}$ for sufficiently large x and x_k-x_j. Details of the asymptotic analysis will be presented elsewhere; here we emphasize two aspects of it: (a) $\langle\mathcal{E}^X\rangle_{\mathcal{B}}^{\mathrm{con}}$ is *dominated* by the *directed* path, i.e., Γ corresponding to $0<x_{i_1}<x_{i_2}<\cdots<x_{i_n}<x$, with all the other paths, containing *overhangs*, being subdominant by a factor of $O(e^{-\Delta x/\xi_{\mathrm{b}}})$ where Δx is the total excess length of the overhangs in the x direction; (b) standard asymptotic analysis of the integrals (18), (22), and (24), reveal that for $x_k-x_j\gg\xi_{\mathrm{b}}$, $y_j=o(x_k-x_j)$ and $w-1$ close to zero when positive, $\mathbf{G}_{jk}$ is dominated *only* by its top-left element, $-G_{jk}^{-+}$, and U_{i_1} and V_{i_n} are dominated by their top elements, $G_{i_1}^{+}(0)$ and $G_{i_n}^{-}(x)$ respectively, with $G_{jk}^{-+}\sim K(x_k-x_j;y_j,y_k)$, $G_{i_1}^{+}(0)\sim K(x_{i_1};0,y_{i_1})$, $G_{i_n}^{-}(x)\sim -K(x-x_{i_n};y_{i_n},0)$ and $G(x)\sim K(x;0,0)$ (ignoring constant prefactors which depend only on the lattice parameters K_1, K_2 and h_1) where $K(\cdot)$ is given by Eq. (14). Hence, from Eq. (21), this implies that $(-1/2)^{|X|}\langle\mathcal{E}^X\rangle_{\mathcal{B}}^{\mathrm{con}}$ asymptotically tends to Eq. (12).

The calculation for the SOS model is simpler. If $\{x_1,\ldots,x_n\}\subset\{1,\ldots,x-1\}$, then the family of finite-dimensional distributions given by $\mathbf{Q}_x([0,\infty)^{1+x}|Y_{x_1}\in A_1,\ldots,Y_{x_n}\in A_n)$ can be *exactly* evaluated using the transfer-integral methods of Ref. [10]. One then applies standard asymptotic methods to the resulting expression for large x with $x_{j+1}-x_j\gg 1$, keeping $a-1/\kappa$ small when positive, leading to the joint probability density function given by Eq. (12), where $K(\cdot)$ is given by Eq. (14).

Thus, for both the Ising and the SOS model we have established the consistent family of finite-dimensional distributions $\{p_{x,n}(\cdot)\}_{n\geq 1}$. By the *Kolmogorov extension theorem* [11] this implies that the measure $\mathbf{P}_x^c$, given by Eq. (7), *uniquely* determines the statistics of the interface on a sufficiently large scale. We now make two remarks.

(i) For $c<0$, the wetting layer thickness is given by $\ell=1/4\tilde{\tau}|c|$ and therefore $2cL_x$ in (7) can be rewritten as $-L_x/2\tilde{\tau}\ell$. From this it follows that the measure $\mathbf{P}_x^c$ is *manifestly* invariant under the scale transformation $\ell\mapsto b\ell$, $x\mapsto b^2x$ and $Y_s\mapsto bY_{b^2s}$.

(ii) The expectation $\lambda:=\lim_{x\to\infty}\mathbf{E}L_x/x$ provides a measure of the average proportion of the substrate staying close to the interface in the thermodynamic limit. It follows from Eqs. (8) and (10) that $2\lambda=\partial\tau^{\times}/\partial c$ from which we have that $\lambda=2\tilde{\tau}|c|$ for $c<0$ and $\lambda=0$ for $c>0$. Hence, we can see that no matter how close one is to the wetting transition for $T<T_w$, some proportion of the interface (which gets vanishingly small as $T\uparrow T_w$) will stay close to the substrate and this *recurrent* property of the interface [16] is not evident from looking at the wetting layer thickness (where $\ell\to\infty$ as $T\uparrow T_w$) alone.

To conclude, we have used *exact* methods to show that wetting in the two-dimensional Ising model is *uniquely* described on sufficiently large length scales by an interface model. *All* aspects of critical wetting in the *asymptotic scaling regime* are contained within this interface model.

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